XXXII. *Of the Theory of Circulating Decimal Fractions.* *By* John Robertſon, *Lib. R. S.*

Read Nov. 24, 1768.

THE great advantages ariſing from the uſe of Arithmetic, particularly in philoſophy and commerce, is ſufficiently known; therefore, every ſtep taken towards its perfection, has always been countenanced by thoſe who were beſt acquainted with its nature and value. On theſe motives I have been induced to offer the annexed paper to this learned Society.

Regiomontanus, it is ſaid, firſt among Europeans, added to the then known arithmetic, an operation by decimal fractions; which he exemplified in his triangular table. Its utility was readily ſeen, and embraced in many nations, and particularly in this; where it appears to have been cultivated in its theory, and facile modes of operation, more than in other places.

Many writers have remarked its excellency in numeral computation, and have pointed out compendiums to avoid the trouble of writing down ſuperfluous figures; particularly in the operations with concrete numbers, or thoſe relative to money, weight, and meaſure; where the gradations from one denomination to another do not proceed in an uniform progreſſion.

In finding the decimal values of the fractional parts of concrete, and other numbers, it often hap-

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pens, that thofe decimals do not terminate, or end, with a few figures only; and fometimes are infinite, or never end; and among thefe are many which have one or more figures conftantly recurring; as in the following proportions, *viz.*

3 : 2 :: 1,0000, *&c.* : 0,6666, *&c.*
and 12 : 5 :: 1,0000, *&c.* : 0,4166, *&c.*
alfo 7 : 3 :: 1,0000, *&c.* : 0,428571,428571, *&c.*

In operations, with fuch recurring decimal fractions, particularly in multiplication and divifion, the work will either be longer than neceffary, or be very inaccurate, if the numbers are not confidered as circulating ones: and to come at the true refults of fuch operations, feveral authors have given precife rules; and fome of them have fhewn the principles upon which thofe rules were founded.

In the annexed paper thofe principles are, endeavoured to be, exhibited in a different, and in a more general and concife manner, than has hitherto been fhewn: but the modes of working are not here annexed, as they are to be found in *Cunn*, *Malcolm*, *Marfh*, and others; and may hereafter be fully exemplified in a treatife of Arithmetic, by the author hereof, confidered in a more mathematical order, than what has hitherto been appropriated to this moft ufeful fcience.

General Principles.

1. Number is fuppofed to begin at unity, and from thence to afcend and defcend: thofe terms afcending above unity, are integers; and thofe defcending below unity, are fractions.

When

When in the aſcending and deſcending parts of the ſcale, the gradation proceeds by a tenfold value from the right hand towards the left, the rank of numbers, thus generated, is called the decimal ſcale.

As every place in this ſcale is ten times the value of its next right hand place; therefore the firſt place in thefractional part, is $\frac{1}{10}$ of the place of units; and the ſecond, third, fourth, *&c.* deſcending places in the fractional part, is $\frac{1}{100}$, $\frac{1}{1000}$, $\frac{1}{10000}$, *&c.* part of the place of units.

Therefore every decimal fraction is equal to a ſeries ariſing from multiplying the firſt, ſecond, third, fourth, *&c.* terms of the decreaſing geometrical progreſſion $\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}$, *&c.* by the firſt, ſecond, third, and fourth, *&c.* terms in the given fraction reſpectively.

Thus. Let the given fraction be 0,3587 or $\frac{3587}{10000}$.

$$\text{Then } 0{,}3587=\frac{1}{10}\times 3+\frac{1}{100}\times 5+\frac{1}{1000}\times 8+\frac{1}{10000}\times 7$$
$$=\frac{3}{10}+\frac{5}{100}+\frac{8}{1000}+\frac{7}{10000}$$
$$=\frac{3000}{10000}+\frac{500}{10000}+\frac{80}{10000}+\frac{7}{10000}=\frac{3587}{10000}.$$

2. Every decimal fraction ariſes from diviſion, when the dividend is leſs than the diviſor.

For, diviſor : dividend :: 10 : firſt term of the fraction;
:: 100 : ſum of the firſt and ſecond terms;
:: 1000 : ſum of the firſt, ſecond, and third, terms, *&c.*

And according to the ratio of the diviſor to the dividend, the quotient, or decimal fraction, will be finite or infinite.

3. Among thoſe decimal fractions which are infinite, or do not end, ſome of them *recur*, or circulate; that is, the ſame figure or figures run over again and again *ad infinitum*.

As 0,333 *&c.*; 0,2323 *&c.*; 0,758758 *&c.*; 0,999 *&c.*

Here $0{,}333\ \&c. = \frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}$, *&c.*

$0{,}2323\ \&c. = \frac{2}{10}+\frac{3}{100}+\frac{2}{1000}+\frac{3}{10000}$, *&c.*

$0{,}785785\ \&c. = \frac{7}{10}+\frac{8}{100}+\frac{5}{1000}+\frac{7}{10000}+\frac{8}{100000}+\frac{5}{1000000}$, *&c.*

$0{,}999,\ \&c. = \frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\frac{9}{10000}$, *&c.*

 The

The circulating fraction 0,999, *&c.* is equal to 1,0.

For the difference between 1,0 and 0,999 *&c.* is leſs than can be aſſigned.

To ſave a repetition of figures, it is uſual to mark the firſt and laſt of circulating expreſſions, with points over the figures.

Thus, 0,333 *&c.* is wrote 0,3
0,2323 *&c.* 0,23
0,785785 *&c.* 0,785

4. Like circulating decimal fractions are thoſe which have each the ſame number of circulating places; and begin to recur each at the ſame name.

Every finite decimal fraction may be conſidered as infinite; cyphers being uſed as the circulating part.

Either place of a circulating expreſſion may be taken as the firſt; obſerving that the number and order of the circulating places be not altered.

For as the decimal fraction ariſes by diviſion; if either place of the recurring figures be taken for the firſt, the others will from thence regularly circulate.

Hence ſeveral unlike circulating decimal fractions may be made to begin and end at places of like names.

5. If in the decimal ſcale 10, 100, 1000, 10000, 100000, 1000000, *&c.* continued indefinitely, be ſelected any rank of equi-diſtant terms, ſuch, that whatever term therein is taken for the firſt term, and the firſt term is made the common ratio to the reſt; then will the ſum of the reciprocals of thoſe terms, be equal to the reciprocal of the number which is unity leſs than the firſt term.

Thus $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \&c. = \frac{1}{9}$; 10 being the 1ſt term.
$\frac{1}{100} + \frac{1}{10000} + \frac{1}{1000000} + \&c. = \frac{1}{99}$; 100 being the 1ſt term.
$\frac{1}{1000} + \frac{1}{1000000} + \frac{1}{1000000000} + \&c. = \frac{1}{999}$; 1000 being the 1ſt term.

For

For $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \&c. = \frac{100+10+1}{1000} = \frac{111\ \&c.}{1000\ \&c.}$

$\frac{1}{100} + \frac{1}{10000} + \frac{1}{1000000} + \&c. = \frac{10000+100+1}{1000000} = \frac{10101\ \&c.}{1000000\ \&c.}$

$\frac{1}{1000} + \frac{1}{1000000} + \frac{1}{1000000000} + \&c. = \frac{1000000+1000+1}{1000000000} = \frac{1001001}{1000000000\ \&c.}$

But $111\ \&c. \times 9 = 1000\ \&c.$ (by 3[d]) Then $\frac{111\ \&c.}{1000\ \&c.} = \frac{1}{9}$

$10101\ \&c. \times 99 = 100000\ \&c.$ Then $\frac{10101\ \&c.}{1000000\ \&c.} = \frac{1}{99}$

$1001001\ \&c. \times 999 = 1000000000\ \&c.$ Then $\frac{1001001\ \&c.}{1000000000\ \&c.} = \frac{1}{999}$

Hence the reciprocal of a number consisting of *n* places of 9's, is equal to a circulating number of *n* places, the right hand figure being 1, and the rest 0's.

Thus, $\frac{1}{9} = \frac{111\ \&c.}{1000\ \&c.} = 0,\dot{1}$

$\frac{1}{99} = \frac{10101\ \&c.}{1000000\ \&c.} = 0,\dot{0}\dot{1}$

$\frac{1}{999} = \frac{1001001\ \&c.}{1000000000\ \&c.} = 0,\dot{0}0\dot{1}$

$\frac{1}{9999} = \frac{10001001\ \&c.}{1000000000000\ \&c.} = 0,\dot{0}00\dot{1}$

6. If the reciprocal of a number consisting of *n* places of 9's, be multiplied by a number D, not exceeding *n* places; the product will be a circulating decimal fraction of *n* places, the right hand ones being the same digits as are in the number D.

Let D = 3; or D = 23; or D = 785; or to any other number.

 Now

Now $\frac{1}{9} = \frac{111 \,\&c.}{1000 \,\&c.}$; $\frac{1}{99} = \frac{10101 \,\&c.}{1000000 \,\&c.}$; $\frac{1}{999} = \frac{1001001 \,\&c.}{1000000000 \,\&c.}$ (by 5th).

Therefore $\frac{1}{9}$, or $\frac{111 \,\&c.}{1000 \,\&c.} \times 3 = \frac{333 \,\&c.}{1000 \,\&c.} = 0{,}\dot{3}$

$$\frac{1}{99}, \text{ or } \frac{10101 \,\&c.}{1000000 \,\&c.} \begin{cases} \times 3 = \frac{30303 \,\&c.}{1000000 \,\&c.} = 0{,}\dot{0}\dot{3} \\ \times 23 = \frac{232323 \,\&c.}{1000000 \,\&c.} = 0{,}\dot{2}\dot{3} \end{cases}$$

$$\frac{1}{999}, \text{ or } \frac{1001001 \,\&c.}{1000000000 \,\&c.} \begin{cases} \times 3 = \frac{3003003 \,\&c.}{1000000000 \,\&c.} = 0{,}\dot{0}0\dot{3} \\ \times 23 = \frac{23023023 \,\&c.}{1000000000 \,\&c.} = 0{,}\dot{0}2\dot{3} \\ \times 785 = \frac{785785785 \,\&c.}{1000000000 \,\&c.} = 0{,}\dot{7}8\dot{5} \end{cases}$$

7. Hence every circulating decimal fraction will be equivalent to a vulgar fraction, wherein the numerator is thoſe circulating figures, and the denominator conſiſts of as many 9's, as are figures in the numerator.

Thus $0{,}\dot{3} = \frac{3}{9}$. For $\frac{1}{9} \times 3 = \frac{3}{9} = 0{,}\dot{3}$ (by 6th)

$0{,}\dot{0}\dot{3} = \frac{3}{99}$ $\quad \frac{1}{99} \times 3 = \frac{3}{99} = 0{,}\dot{0}\dot{3}$

$0{,}\dot{2}\dot{3} = \frac{23}{99}$ $\quad \frac{1}{99} \times 23 = \frac{23}{99} = 0{,}\dot{2}\dot{3}$

8. Hence a circulating decimal fraction, of any number of places, being multiplied by a number of as many 9's, will give a finite expreſſion, having the ſame figures as are in the circulating one.

Thus $0{,}\dot{6} \times 9 = 6$. For $9 : 1 :: 6 : 0{,}\dot{6}$

$0{,}\dot{0}\dot{6} \times 99 = 6$. $\quad 99 : 1 :: 6 : 0{,}\dot{0}\dot{6}$

$0{,}\dot{2}\dot{5} \times 99 = 25$. $\quad 99 : 1 :: 25 : 0{,}\dot{2}\dot{5}$

$0{,}\dot{6}2\dot{5} \times 999 = 625$. $\quad 999 : 1 :: 625 : 0{,}\dot{6}2\dot{5}$

Hence

Hence it appears, that, in common multiplication, the product of a circulating number, by its proper denominator, in 9's, will be deficient of the true product by that circulating number.

Thus $0,\dot{6} \times 9 = 5,4$; then $5,4 + ,6 = 6$. For $\frac{6}{9} = 0,\dot{6}$

$0,0\dot{6} \times 99 = 5,94$; then $5,94 + 0,06 = 6$. For $\frac{6}{99} = 0,0\dot{6}$

$0,\dot{6}2\dot{5} \times 999 = 624,375$; then $624,375 + 0,625 = 625,0$

Hence. Any finite number is in proportion to the ſame number recurring, as the proper denominator of the circulate is to that denominator increaſed by unity.

Thus $9 : 10 :: 6 : \dot{6}$. For $\dot{6} \times 9 = 6 \times 10$

$99 : 100 :: 25 : \dot{2}\dot{5}$. For $\dot{2}\dot{5} \times 99 = 25 \times 100$.

SCHOLIUM.

If to the preceding articles, be joined the compendiums of multiplying and dividing by any number of 9's, they will conſtitute the whole of the theory, upon which depend all the operations with circulating numbers: for as theſe have 9's for their denominator, wanting unity in the loweſt place to make them 10's; therefore unity for every 9 is applied in ſome additions and multiplications: Or, the circulating parts being reduced to finite number; then working with them by the common rules, will give finite reſults; which reſults are to be reduced to circulates by contrary operations to what were uſed to reduce the circulates to finites.